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The t-invariant of analytic set germs of dimension 2 ☆

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Abstract

Let $X_0 \subset \mathbb{R}^n$ be an analytic set germ of dimension 2. We study the invariant $t(X_0)$ defined as the least integer t such that any open semianalytic set germ of X_0 can be written as a union of t basic open set germs. It is known that $2 \leq t(X_0) \leq 3$. In this note we provide a geometric criterion to determine the exact value of $t(X_0)$. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let X_0 denote a real analytic set germ of dimension two. We can suppose X_0 to be a subgerm of $\mathbb{R}^n$ at the origin. We denote by $\mathcal{O}(X_0) = \mathcal{O}_n/\mathcal{J}(X_0)$ the ring of analytic function germs at X_0, where $\mathcal{O}_n$ represents the ring of analytic function germs at $0 \in \mathbb{R}^n$ and $\mathcal{J}(X_0)$ stands for the ideal of function germs $f \in \mathcal{O}_n$ which vanish at X_0. From now on we will suppose that X_0 is irreducible, that is, the ideal $\mathcal{J}(X_0)$ is prime, and denote by $\mathbb{K}(X_0)$ the quotient field of $\mathcal{O}(X_0)$.

A *basic open semianalytic set germ* of X_0 is a set germ of the form $S_0 = \{g_1 > 0, \ldots, g_r > 0\} \subset X_0$, where $g_i \in \mathcal{O}(X_0)$. *Basic closed semianalytic set germs* are defined in the same way relaxing inequalities. The *stability index* $s(X_0)$ (resp. *closed stability index* $\bar{s}(X_0)$) is defined to be the least integer s such that any basic open (resp. closed) semianalytic set germ of X_0 can be written with s elements of $\mathcal{O}(X_0)$. It is known [1, Theorem VIII.2.12] that

$$s(X_0) = 2 \quad \text{and} \quad 2 \leq \bar{s}(X_0) \leq 3.$$

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Moreover, it is shown in [5] that both values are attained.

The *semianalytic set germs* are finite boolean combinations of basic open semianalytic set germs. By the Finiteness Theorem [1, Corollary VIII.3.2] every open (resp. closed) semianalytic set germ can be written as a union of a finite number of basic open (resp. closed) semianalytic set germs. We define $t(X_0)$ (resp. $\bar{t}(X_0)$) as the minimum integer t such that any open (resp. closed) semianalytic set germ can be written as a union of t basic open (resp. closed) semianalytic set germs. In [1, Proposition V.2.16 and Theorem VIII.2.12] it is shown that

$$2 \leq t(X_0) \leq 3 \quad \text{and} \quad \bar{t}(X_0) = 2.$$

In this note we show that $t(X_0) = \bar{s}(X_0)$, so that we have a complete geometric characterization of two-dimensional germs having $t(X_0) = 3$, cf. Theorem 1.2 below.

1. Preliminaries

Let $S \subset X_0$ be a semianalytic set germ. We denote by $\operatorname{Adh} S$, $\operatorname{Int} S$ and $\operatorname{Bd} S$ the closure, interior and boundary of S, respectively. $\bar{S}^Z$ will denote the Zariski closure of S. In dealing with set germs we will often omit any reference to the origin. Two semianalytic set germs are said to be *generically equal* if they equal each other up to a semianalytic subset of codimension one.

Let $\operatorname{Spec}_r \mathcal{O}(X_0)$ be the real spectrum of $\mathcal{O}(X_0)$, cf. [3, Chapter 6]. Then, to any semianalytic set germ S of X_0 we attach the constructible set $\tilde{S}$ of $\operatorname{Spec}_r \mathcal{O}(X_0)$ defined by the same formula. This *tilde map* defines an isomorphism from the lattice of semianalytic set germs of X_0 to the lattice of constructible sets of $\operatorname{Spec}_r \mathcal{O}(X_0)$, cf. [1, Theorem VIII.2.5]. Moreover, it gives a one-to-one correspondence between the constructible points of $\operatorname{Spec}_r \mathcal{O}(X_0)$ and half-branches of irreducible analytic curves on X_0.

We will need the Hörmander–Łojasiewicz inequality in the following practical form, cf. [5] (see also [3, Lemma 7.7.10] for the semialgebraic analogous):

Proposition 1.1 (Hörmander–Łojasiewicz inequality). *Let $T \subset X_0$ be a closed semianalytic germ and $f, g \in \mathcal{O}(X_0)$. Then there exist $p, q \in \mathcal{O}(X_0)$ such that*
(a) $p > 0$, $q \geq 0$ *on* X_0,
(b) $\operatorname{sign}(pf + qg) = \operatorname{sign}(f)$ *over* T,
(c) $\{q = 0\} \subset \overline{\{f = 0\} \cap T}^Z$.

Let $\pi : X_0^\nu \to X_0$ be the *normalization* of X_0. The corresponding homomorphism $\pi^* : \mathcal{O}(X_0) \to \mathcal{O}(X_0^\nu)$ is injective and induces an isomorphism between their quotient fields $\mathbb{K}(X_0)$ and $\mathbb{K}(X_0^\nu)$. For generalities on normalization we refer to [6, Chapter VI]; [8, Chapter III]. We have the following geometric characterization of the values of $\bar{s}$ in the two-dimensional case, cf. [4]:

Theorem 1.2. *Let X_0 be an analytic set germ of dimension two and $\pi : X_0^{\nu} \to X_0$ its normalization. Then the following conditions are equivalent*:

(1) *There exists an irreducible analytic curve germ $Y \subset X_0^{\nu}$ such that $\pi(Y)$ is a unique half-branch.*

(2) $\bar{s}(X_0) = 3$.

(3) *There exists a 4-element fan $F \subset \mathrm{Spec}_r\, \mathcal{O}(X_0)$ specializing to a unique ordering $\tilde{\tau} \in \mathrm{Spec}_r\, \mathcal{O}(X_0)$ $(F \to \tilde{\tau})$ with* $\dim \tilde{\tau} = 1$.

In particular, if X_0 is normal then $\bar{s}(X_0) = 2$. We illustrate the above result with a typical example.

Example 1.3. Let X_0 be *Whitney's umbrella*, that is the germ of zeroes of $\mathrm{y}^2 - \mathrm{z}\mathrm{x}^2 \in \mathbb{R}\{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$, see Fig. 1. The normalization of X_0 is the germ X_0^{ν} : $\mathrm{z}' - \mathrm{y}^{2'} = 0$, where $\mathrm{x} = \mathrm{x}', \mathrm{y} = \mathrm{x}'\mathrm{y}', \mathrm{z} = \mathrm{z}'$ (Fig. 2).

The parabola $Z' = X_0^{\nu} \cap \{\mathrm{x}' = 0\}$ corresponds to the positive z-axis in X_0, that is, $\pi(Z')$ is a single half-branch. On the other hand, $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is the pull-back of the trivial fan $\{\tilde{\tau}_1, \tilde{\tau}_2\}$ defined by the half-branches of Z' at the origin, via the discrete valuation ring $\mathcal{O}(X_0^{\nu})_{\mathfrak{q}}$ where $\mathfrak{q} = \mathcal{J}(Z')$ and $F \to \tilde{\tau}$, cf. [2, Chapter 7]. Thus conditions (1) and (3) of the Theorem 1.2 are fulfilled, so $\bar{s}(X_0) = 3$.

Using the above result it is easy to show that $\bar{s}(X_0) = 3$ implies $t(X_0) = 3$. Before showing this we recall, cf. [1, Corollary V.1.8], that an open semianalytic germ $B \subset X_0$ is basic if and only if it verifies that: (a) $B \cap \overline{\mathrm{Bd}(B)}^Z = \emptyset$ (i.e., the Zariski closure of the boundary of B does not reenter in B), and (b) $\#(F \cap \tilde{B}) \neq 3$ for any 4-element fan $F \subset \mathrm{Spec}_r\, \mathcal{O}(X)$.

Proposition 1.4. *If* $\bar{s}(X_0) = 3$, *then* $t(X_0) = 3$.

Proof. Since $\bar{s} = 3$, there is a 4-element fan F specializing to $\tilde{\tau} \in \mathrm{Spec}_r\, \mathcal{O}(X_0)$ with $\dim \tilde{\tau} = 1$. Let us denote by τ the half-branch corresponding to $\tilde{\tau}$ and set $\bar{\tau}^Z = \tau \cup \tau'$. Consider an open semianalytic set germ C such that $\tau' \subset C$, $\tau \not\subset C$ and $\#(\tilde{C} \cap F) = 3$. For instance, if $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and f_1, f_2, $f_3 \in \mathcal{O}(X_0)$ are such that $\{\alpha_1, \alpha_2\} \subset \{f_1 < 0\}$,

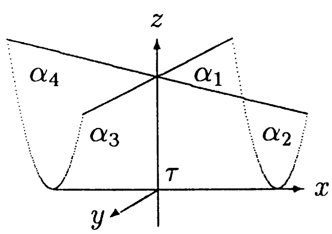

Fig. 1.

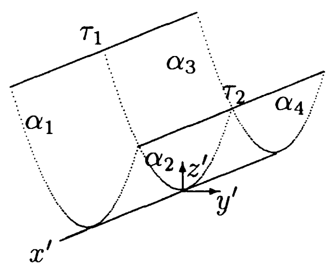

Fig. 2.

$\{\alpha_3,\alpha_4\} \subset \{f_1>0\}$, $\alpha_1 \in \{f_2<0\}$, $\alpha_2 \in \{f_2>0\}$, $\tilde{\tau} \in \{f_3<0\}$, $\tilde{\tau}' \in \{f_3>0\}$, then, we can take $C=\{f_1>0\}\cup\{f_2>0\}\cup\{f_3>0\}$.

Now, suppose we can write $C=B_1\cup B_2$, with B_1 and B_2 basic open sets. As $\tau' \subset C$ we can suppose $\tau' \subset B_1$. B_1 cannot be adherent to τ since otherwise, B_1 will intersect its Zariski boundary and it would not be basic. So the three elements of F included in $\tilde{C}$ must belong to $\tilde{B}_2$, but again this is not possible for B_2 is basic. Thus C cannot be the union of two basic sets and so $t(X_0)=3$. □

Example 1.5. Let X_0 be again *Whitney's umbrella*. By the previous proposition it must be $t(X_0)=3$. Moreover, following the proof it can be shown that the open semianalytic set germ $C=\{x<0\}\cup\{y<0\}\cup\{z<0\}$ cannot be written as the union of two basic sets. In this case, τ and τ' are, respectively, the positive and negative z-axis. Thus, with the notation of Example 1.3, $\tilde{C}\cap F=\{\alpha_1,\alpha_3,\alpha_4\}$, $\tau' \subset C$ and $\tau \not\subset C$. Also, it can be shown, cf. [5], that the complement of C in X_0, the basic closed set $B=\{x\geq 0, y\geq 0, z\geq 0\}$, cannot be written with two inequalities.

It remains to prove $\bar{s}(X_0)=2$ implies $t(X_0)=2$, but this is not as direct as the previous result.

In the following B, B_i, B_i', B_i'' will always denote basic open set germs and γ, γ' the half-branches of one irreducible curve set germ.

Consider an open semianalytic set germ $C\subset X_0$. The closure of C is a closed semianalytic set germ [1, Corollary VIII.3.2] and (being $\bar{t}=2$) it can be written as $\bar{C}=C_1\cup C_2$ with C_1 and C_2 basic closed sets. If $C_i=\{f_i\geq 0, g_i\geq 0, h_i\geq 0\}$, then we define $B_i=\{f_i>0, g_i>0, h_i>0\}$. Thus C and $B_1\cup B_2$ are generically equal, so they differ in a finite number of half-branches. Thus, in order to write C as a union of two basic open sets we will perform two operations;

(1) to cut off half-branches, that is, if $\gamma\not\subset C$ but $\gamma\subset B_1\cup B_2$, to find B_1' and B_2' such that $B_1'\cup B_2'=(B_1\cup B_2)\setminus\bar{\gamma}^Z$, and
(2) to add half-branches, that is, if $\gamma\subset C$, to find B_1' and B_2' such that $B_1'\cup B_2'=B_1\cup B_2\cup\gamma$. Notice that since $B_1'\cup B_2'$ is open, we must have $\gamma\subset \mathrm{Int}\,(\overline{B_1\cup B_2})$.

The procedure will be, first of all, to cut off the half-branches of $(B_1\cup B_2)\setminus C$. Notice that in this proccess we may take out more half-branches than needed, since we are cutting off $\bar{\gamma}^Z$ and not only γ. Thus, we will find $(B_1'\cup B_2')\subset C$ such that $C\setminus(B_1'\cup B_2')=\bigcup_{i=1}^r \gamma_i$. Finally, we will add these half-branches to get $C=B_1''\cup B_2''$.

While adding half-branches requires the hypothesis $\bar{s}(X_0)=2$, cutting off is always possible, since we can always multiply by a positive equation of the curve.

Proposition 1.6. *If $\gamma\not\subset C$ but $\gamma\subset B_1\cup B_2$, then there exist B_1' and B_2' such that $B_1'\cup B_2'=(B_1\cup B_2)\setminus\bar{\gamma}^Z$.*

Proof. We can suppose $B_1=\{f_1>0,\ g_1>0\}$ and $B_2=\{f_2>0, g_2>0\}$. Let $r\in\mathcal{O}(X)$ be a positive equation of $\bar{\gamma}^Z$. Then $B_1'=\{f_1r>0, g_1>0\}$ and $B_2'=\{f_2r>0, g_2>0\}$ satisfy the condition. □

2. The t-invariant in the normal case

In this section we are to see that if X_0 is normal $t(X_0)=2$. Then we will use this result and the normalization of X_0 to get the final result: $\bar{s}=2 \Leftrightarrow t=2$. We start with the following easy observation which will be used in the sequel; here, as well as in Lemmas 2.2–2.4 normality is not required.

Remark 2.1. Let $B \subset X_0$ be a basic open set germ and γ a half-branch satisfying
(i) $\gamma \subset \operatorname{Int} \bar{B}$,
(ii) $\bar{\gamma}^Z \cap \bar{B} = \gamma$,
then $B' = B \cup \gamma$ is a basic open set germ. Indeed, as B is an open semianalytic, $\operatorname{Int} \bar{B}$ and B differ in a finite number of half-branches, say, $\operatorname{Int} \bar{B} = B \cup \gamma \cup (\bigcup_1^r \gamma_i)$. As $\gamma \subset \operatorname{Int} \bar{B}$, we can take a neigborhood V of γ such that $V \subset \operatorname{Int} \bar{B}$ and $V \cap (\bigcup_1^r \gamma_i) = \emptyset$, that is, $V \subset B \cup \gamma$ and so, $B' = B \cup \gamma$ is open. Also, by condition (ii) B' does not intersect its Zariski boundary and being generically equal to B, verifies automatically the 4-element fan condition. Hence B' is basic open.

In the same way, it can be shown that if $\bar{\gamma}^Z \subset \operatorname{Int} \bar{B}$ then $B' = B \cup \bar{\gamma}^Z$ is basic open.

Here are some useful technical lemmata which provide a sort of surgery tools to manipulate half-branches. The first lemma will allow us to take off a neighborhood of a single half-branch from a basic set preserving basicness.

Lemma 2.2. *Let $B \subset X_0$ be a basic open set, γ a half-branch such that $\gamma \subset \bar{B}$ and S a closed semianalytic set germ such that $\gamma \not\subset S$. Then there exists a basic open $B' \subset B$ such that $\gamma \not\subset \overline{B'}$ and $B' \cap S = B \cap S$.*

Proof. There exists, cf. [7], $h \in \mathcal{O}(X_0)$ which separates γ from S, that is, $\gamma \subset \{h < 0\}$ and $S \subset \{h > 0\}$. Thus, $B' = B \cap \{h > 0\}$ is the basic open we were looking for. $\square$

Our next lemma is somehow the opposite of the previous one and will be used to add a neigborhood of the two half-branches of an irreducible curve to a basic set also preserving basicness.

Lemma 2.3. *Let $B \subset X_0$ be a basic open, γ a half-branch and S a closed semianalytic set germ such that $\bar{\gamma}^Z \cap S = \emptyset$. Then there exist B' basic open such that $B \cup \bar{\gamma}^Z \subset B'$ and $B' \cap S = B \cap S$.*

Proof. Set $B = \{f > 0, g > 0\}$. Multiplying f and g by a positive equation of $\bar{\gamma}^Z$ we may assume, without loss of generality, that $f = g = 0$ on $\bar{\gamma}^Z$. Now, take any h such that $B \cup \bar{\gamma}^Z \subset \{h > 0\}$ (for instance $h = 1$), and apply Hormänder–Łojasiewicz inequality to f, h , S and to g, h, S. Then $B' = \{pf + qh > 0, p'g + q'h > 0\}$ does the job. Indeed, $B \cap S = B' \cap S$ by construction. Also, $\{q = 0\} \subset \overline{\{f = 0\} \cap S}^Z$ and since $\bar{\gamma}^Z \cap S = \emptyset$ we have that $q|_{\bar{\gamma}^Z} > 0$ and then $pf + qh|_{\bar{\gamma}^Z} > 0$. $\square$

This lemma can be refined in case one half-branch does not intersect S.

Lemma 2.4. *Let $B \subset X_0$ be a basic open, γ a half-branch and S a closed semianalytic set germ such that $\gamma \cap S = \emptyset$. If $\bar{\gamma}^Z = \gamma \cup \gamma'$ and $\gamma' \not\subset \bar{B}$, then there exist B' basic open such that $B \cup \gamma \subset B'$ and $B' \cap S = B \cap S$.*

Proof. Set $B = \{f > 0, g > 0\}$ and assume $f = g = 0$ on $\bar{\gamma}^Z$ as in the previous lemma. Also, set $\Gamma = \{fg = 0\}$ and take $h \in \mathcal{O}(X_0)$ such that $h|_\gamma < 0$ and $h|_{S \cup \gamma'} > 0$. Let U_γ, $U_{\gamma'}$ be connected open semianalytic neighborhoods of γ and γ', respectively such that $\overline{U_\gamma} \cap [S \cup \Gamma \cup \{h = 0\}] \subset \gamma$ and $\overline{U_{\gamma'}} \cap [\Gamma \cup \{h = 0\}] \subset \gamma'$. Now we will change f and g by f' and g' such that they had equal sign in a neighborhood of γ. We apply Hörmander–Łojasiewicz to f, $-h$, $[X_0 \setminus (U_\gamma \cup U_{\gamma'})]$ and set $f' = pf - qh$. By construction, $f'|_\gamma > 0$, $\operatorname{sign}(f') = \operatorname{sign}(f)$ at least on $[X_0 \setminus (U_\gamma \cup U_{\gamma'})]$ and then also they have equal sign on $S \setminus U_{\gamma'}$ (since $S \subset X_0 \setminus U_\gamma$). The same can be done with g to obtain $g' = p'g - q'h$ such that $g'|_\gamma > 0$ and $\operatorname{sign}(g') = \operatorname{sign}(g)$ on $S \setminus U_{\gamma'}$. Thus we have $\gamma \subset \{f' > 0, g' > 0\}$ and $\{f' > 0, g' > 0\} \cap (S \setminus U_{\gamma'}) = B \cap (S \setminus U_{\gamma'})$. Further, since $U_{\gamma'} \cap B = \emptyset$ and $h|_{U_{\gamma'}} > 0$ it can be checked that $\{f' > 0, g' > 0\} \cap U_{\gamma'} = B \cap U_{\gamma'} = \emptyset$ for if we restrict to $U_{\gamma'}$ we have $h > 0$ and so $f' > 0$, $g' > 0$ implies $f > 0$, $g > 0$. Now, we have $\gamma \subset \{f' > 0, g' > 0\}$, $\{f' > 0, g' > 0\} \cap S = B \cap S$ and $\gamma' \not\subset \overline{\{f' > 0, g' > 0\}}$ (because $f'|_{\gamma'} < 0$, $g'|_{\gamma'} < 0$), so we can take $B' = \{f' > 0, g' > 0\}$. $\square$

We can modify slightly this lemma multiplying by a positive equation of γ to obtain B' such that $\gamma \subset \operatorname{Int} \overline{B'}$ but $\gamma \not\subset B'$. The same modification can be done with Lemma 2.3 to obtain B' such that $\bar{\gamma}^Z \subset \operatorname{Int} \overline{B'}$ but $\gamma, \gamma' \not\subset B'$. Let us see with an easy example how these lemmas work.

Example 2.5. Let $X_0 = \mathbb{R}^2$ and $B = \{2x - y > 0, y > 0\}$ (see Fig. 3). We can take off a neighborhood of γ_1 (the positive x-axis) from B using Lemma 2.2. Also we can use Lemma 2.4 to can add a neighborhood of γ_2 (the positive y-axis) to B. More precisely,

(a) if $h = x^3 - y^2$ then $B' = B \cap \{h > 0\}$ (see Fig. 4) is not adherent to γ_1 and $B' \cap S = B \cap S$ for S as big as $\{h \leq 0\}$. Of course, if S is bigger we will have to take h with zero set closer to γ_1.

(b) Following Lemma 2.4, we write $B = \{(2x - y)x^2 > 0, yx^2 > 0\}$ and then taking $f' = (2x - y)x^2 - (2x - y)^2(x^2 - y^3)$, $g' = yx^2 - (x^2 - y^3)y^2$ we add a neighborhood of γ_2 to B obtaining $B'' = \{f' > 0, g' > 0\}$ (see Fig. 5). As before bigger S implies smaller neighborhood of γ_2.

We say that f *changes sign along* a half-branch γ if any representative of γ in any neigborhood of the origin contains points in which $f > 0$ and points in which $f < 0$. In case X_0 is a normal germ and $\bar{\gamma}^Z = \gamma \cup \gamma'$ if f changes sign along γ then also changes sign along γ', cf. [5]. We recall also that if X_0 is normal then X_0 is non-singular in codimension one, that is, the origin is at most the unique singular point of X_0 and, in particular, X_0 is of pure dimension.

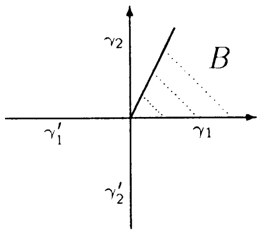

Fig. 3.

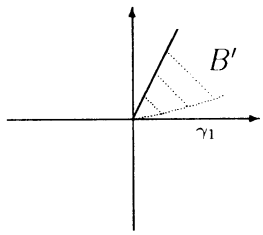

Fig. 4.

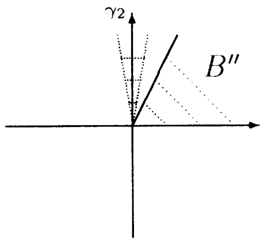

Fig. 5.

Lemma 2.6. *Let X_0 be a normal analytic set germ, $B \subset X_0$ a basic open set and $\gamma \subset \mathrm{Bd}(\bar{B})$. We can write $B = \{f > 0, g > 0\}$ in such a way that only one of f or g, changes sign along γ.*

Proof. Suppose $B = \{f_1 > 0, f_2 > 0\}$, with f_1 and f_2 changing sign along γ. Then $f_1 f_2$ does not change sign along γ and $B = \{f_1 > 0, f_2 > 0\} = \{f_1 f_2 > 0, f_2 > 0\}$, we are done. $\square$

Proposition 2.7. *Let B_1, B_2 be basic open set germs and γ a half-branch. If X_0 is normal and $\gamma \subset \mathrm{Int}(\overline{B_1 \cup B_2})$, then there exist B_1' and B_2' basic open such that $B_1 \cup B_2 \cup \gamma = B_1' \cup B_2'$.*

Moreover, if S is a closed semianalytic set germ such that $\bar{\gamma}^Z \cap S = \{0\}$, then we can take B_1', B_2' such that $B_i' \cap S = B_i \cap S$, $i = 1, 2$. Also, if $\bar{\gamma}^Z = \gamma \cup \gamma'$, then $\gamma \subset B_1'$, $\gamma' \not\subset \overline{B_1'}$.

Proof. Set $B_1 = \{f_1 > 0, g_1 > 0\}$, $B_2 = \{f_2 > 0, g_2 > 0\}$ and $\Gamma = \{f_1 g_1 f_2 g_2 = 0\}$. By Lemma 2.6 we can suppose f_1 and f_2 do not change sign along γ. We take U_γ, $U_{\gamma'}$ as connected open semianalytic neighborhoods of γ and γ', respectively, such that $\overline{U_\gamma} \cap [S \cup \Gamma \cup \{h = 0\}] \subset \gamma$ and $\overline{U_{\gamma'}} \cap [S \cup \Gamma \cup \{h = 0\}] \subset \gamma'$. We distinguish two cases:

1. $\gamma' \subset \mathrm{Int}\,\overline{B_1 \cup B_2}$. As $\bar{\gamma}^Z \subset \overline{B_1 \cup B_2}$ we have $\overline{U_\gamma} \subset (B_1 \cup B_2 \cup \gamma)$ and $\overline{U_{\gamma'}} \subset (B_1 \cup B_2 \cup \gamma')$. Applying Lemma 2.3 to B_1, γ and $[X_0 \setminus (U_\gamma \cup U_{\gamma'})]$ we can suppose $\bar{\gamma}^Z \subset \mathrm{Int}\,\overline{B_1}$. In the same way it can be supposed $\bar{\gamma}^Z \subset \mathrm{Int}\,\overline{B_2}$.

In this situation we apply now Lemma 2.2 to B_1, γ', $[X_0 \setminus U_{\gamma'}]$ on the one hand and to B_2, γ, $[X_0 \setminus U_\gamma]$ on the other. By doing this we find B_1^1, B_2^1 such that $\gamma' \not\subset \overline{B_1^1}$, $\gamma \subset \operatorname{Int} \overline{B_1^1}$, $\gamma' \subset \operatorname{Int} \overline{B_2^1}$, $\gamma \not\subset \overline{B_2^1}$ and so we can take $B_1' = B_1^1 \cup \gamma$, cf. Remark 2.1, and $B_2' = B_2$ (or $B_2' = B_2^1 \cup \gamma'$ in case $\gamma' \subset B_1 \cup B_2$). By construction $(U_\gamma \cup U_{\gamma'}) \cap S = \emptyset$ so $B_i' \cap S = B_i \cap S$ and we are done.

2. $\gamma' \not\subset \operatorname{Int} \overline{B_1 \cup B_2}$. If $\gamma' \not\subset \overline{B_1}$ then applying Lemma 2.4 to B_1, γ, $[X_0 \setminus U_\gamma]$ we are done. So we can suppose that $\gamma' \subset \overline{B_1}$ and more precisely that $\gamma' \subset \operatorname{Bd} \overline{B_1}$. Thus by applying Lemma 2.2 to B_2, γ' and $[X_0 \setminus U_{\gamma'}]$ we can suppose $\gamma' \not\subset \overline{B_2}$. Now, B_2 being so, we apply Lemma 2.4 to B_2, γ and $[X_0 \setminus U_\gamma]$ in order to add a neighborhood of γ to B_2. Finally, we take off a neighborhood of γ to B_1 using Lemma 2.2. Thus, we can reduce to the following situation (see Fig. 6): $\gamma' \subset \operatorname{Bd} \overline{B_1}$, $\gamma \not\subset \overline{B_1}$, $\gamma' \not\subset \overline{B_2}$ and $\gamma \subset \operatorname{Int} \overline{B_2}$. Also, if $B_1 = \{f_1 > 0, g_1 > 0\}$ and $B_2 = \{f_2 > 0, g_2 > 0\}$, we can suppose f_2, g_2 do not change sign along $\bar{\gamma}^Z$ (because $\gamma \subset \operatorname{Int} \overline{B_2}$), $f_2|_{U_{\gamma'}} \le 0$ (because $\gamma' \not\subset \overline{B_2}$), $g_2|_{U_{\gamma'}} \ge 0$ (multiplying g_2 if necessary by a function negative in γ' but positive in $\{f_2 > 0\}$) and f_1 does not change sign along $\bar{\gamma}^Z$ (by Lemma 2.6) but g_1 does (as $\gamma' \subset \operatorname{Bd} \overline{B_1}$).

Now, let $h \in \mathcal{O}(X_0)$ be such that $h|_{\gamma'} < 0$ and $h|_\gamma > 0$. Setting U_γ, $U_{\gamma'}$ as before but redefining Γ as $\{f_1 g_1 f_2 g_2 h = 0\}$, we apply Hörmander–Łojasiewicz to g_1, h, $T = [X_0 \setminus (U_\gamma \cup U_{\gamma'})]$ and set $g_1' = pg_1 + qh$. If we define $B_1^1 = \{f_1 g_1 g_1' > 0, g_1' > 0\}$ and $B_2^1 = \{f_2 g_1 g_1' > 0, g_2 > 0\}$ (see Fig. 7), then it can be checked that $B_1 \cup B_2$ and $B_1^1 \cup B_2^1$ are generically equal and more precisely $(B_1 \cup B_2) \cap \{g_1 g_1' \neq 0\} = (B_1^1 \cup B_2^1) \cap \{g_1 g_1' \neq 0\}$. We have $B_1^1 \cap T = B_1 \cap T$ and $B_2 \cap T = (B_2^1 \cap T) \cup (\{g_1 = 0\} \cap B_2 \cap T)$ (g_1 and g_1' have the same sign on T) so we could have lost some half-branches but $B_2^2 = B_2^1 \cup (\{g_1 = 0\} \cap B_2 \cap T)$ is a basic open set by Remark 2.1. Now, we have $\gamma' \not\subset \overline{B_1^1}$ so applying Lemma 2.4 to B_1^1, γ and $X_0 \setminus U_\gamma^1$ (taking as U_γ^1 a connected open semianalytic neighborhood of γ such that $\overline{U_\gamma^1} \cap [S \cup \Gamma \cup \{g_1' = 0\}] \subset \gamma$) we get B_1^2 basic open such that $\gamma \subset B_1^2$ and $\gamma' \not\subset \overline{B_1^2}$.

It only remains to see what happen with the missing half-branches in $\{g_1 g_1' = 0\} \cap (U_\gamma \cup U_{\gamma'})$. We have $(B_1 \cup B_2) \setminus (B_1^2 \cup B_2^2) \subset \{g_1 g_1' = 0\}$ they differ in a finite number of half-branches, say $\bigcup_1^r \gamma_i \subset U_\gamma \cup U_{\gamma'}$. But, by construction, if $\bar{\gamma}_i^Z = \gamma_i \cup \gamma_i'$, then we have $\gamma_i' \subset \operatorname{Int} \overline{B_1^2 \cup B_2^2}$ (in fact, $\gamma_i' \subset U_\gamma \cup U_{\gamma'}$) and so they can be added as in case 1. $\square$

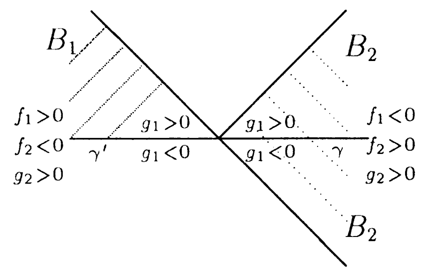

Fig. 6.

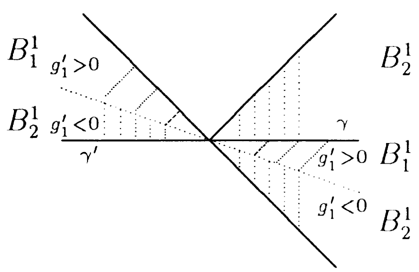

Fig. 7.

Finally, joining Propositions 1.6 and 2.7 we get at once

Corollary 2.8. *If* X_0 *is normal, then* $t(X_0) = 2$.

Remark 2.9. Notice that normality has only been used for the sign property described in Lemma 2.6. Thus, let define a germ X_0 to verify the *change sign property* if it is of pure dimension and for every half-branch γ if $f, g \in \mathcal{O}(X_0)$ change sign along γ the product fg does not change sign along γ.

Obviously, the class of germs verifying the change sign property is wider than that of normal germs and it follows that Proposition 2.7 and Corollary 2.8 above hold for germs with the change sign property.

We illustrate the whole proccess with an example.

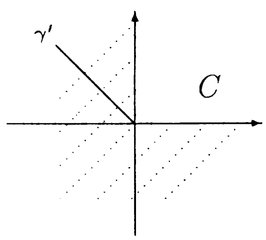

Fig. 8.

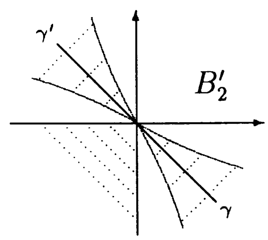

Fig. 9.

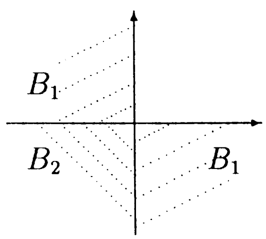

Fig. 10.

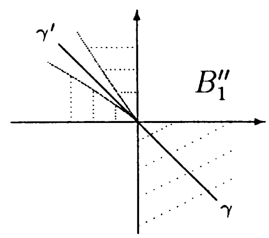

Fig. 11.

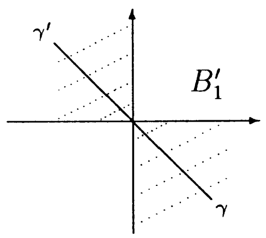

Fig. 12.

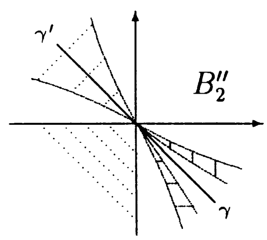

Fig. 13.

Example 2.10. Suppose $X_0 = \mathbb{R}^2$ and $C = (\{-x > 0\} \cup \{-y > 0\}) \setminus (\gamma' \cup \{xy = 0\})$ where γ' is the half-branch defined as $\{x + y = 0, y > 0\}$ (see Fig. 8). $\bar{C}$ is a closed set germ and then it can be written as the union of two basic closed sets, for instance, $\bar{C} = \{-xy \geq 0\} \cup \{-x \geq 0, -y \geq 0\} = C_1 \cup C_2$.

If we define $B_1 = \{-xy > 0\}$ and $B_2 = \{-x > 0, -y > 0\}$, then $B_1 \cup B_2$ and C are generically equal (see Fig. 9). In fact, $(B_1 \cup B_2) \setminus C = \gamma'$ so we must take this half-branch out of $B_1 \cup B_2$ in order to write C as a union of two basic open sets. First, we multiply by $(y+x)^2$ to obtain $B_1' = \{-(y+x)^2xy > 0\}$ (see Fig. 10). Now, we must add γ.

Following Proposition 2.7 we add a neigborhood of $\bar{\gamma}^Z$ to B_2 taking $B_2' = \{-xh_1 > 0, -yh_2 > 0\}$, where $h_1 = (x-y)^2 - (x+y)^3$ and $h_2 = (x-y)^2 + (x+y)^3$ (see Fig. 11). After this, we take off a neighborhood of γ' from B_1' (of course, this neighborhood must be contained in the neighborhood previously added to B_2) obtaining $B_1'' = \{-(y+x)^2xyh_3 > 0\}$ with $h_3 = (x-y)^2 + (x+y)^5$ (see Fig. 12) and also we take off a neigborhood of γ from B_2' obtaining $B_2'' = \{-xh_1h_4 > 0, -yh_2h_4 > 0\}$ with $h_4 = (x-y)^2 - (x+y)^5$ (see Fig. 13). Now, $B_1'' \cup \gamma = B_1'''$ is a basic open set (it can be written as $\{-xyh_3 > 0\}$) and so $C = B_1''' \cup B_2''$ is the decomposition we sought.

3. The *t*-invariant in the general case

Let us consider now an analytic set germ X_0 of dimension two such that $\bar{s}(X_0) = 2$. Let $\pi : X_0^{\nu} \to X_0$ be the normalization of X_0, $C \subset X_0$ an open semianalytic set germ and $C^{\nu} = \pi^{-1}(C) \subset X_0^{\nu}$. We have $C^{\nu} = B_1^{\nu} \cup B_2^{\nu}$ (since $t(X_0^{\nu}) = 2$ by the preceding corollary) and each B_i^{ν} can be written as $B_i^{\nu} = \{f_i' > 0, g_i' > 0\}$. Now, if $f_i' = f_i^*/h^* = f_i \circ \pi / h \circ \pi = \pi^*(f_i/h)$, $g_i' = \pi^*(g_i/h)$, we define $B_i = \{f_ih > 0, g_ih > 0\}$. Thus, C is generically equal to $B_1 \cup B_2$. If we can add to $B_1 \cup B_2$ the half-branches in $C \setminus (B_1 \cup B_2)$ without incrementing the number of basic sets (to cut off is always possible as seen in Proposition 1.6), then C could be written as the union of two basic open sets.

Lemma 3.1. *Let γ be a half-branch and $\bar{\gamma}^Z = \gamma \cup \gamma'$. If $B_i^{\nu} \cup \pi^{-1}(\gamma)$ is basic open and $\pi^{-1}(\gamma') \cap \overline{B_i^{\nu}} = \emptyset$, then $B_i \cup \gamma$ is basic open.*

Proof. $B_i^{\nu} \cup \pi^{-1}(\gamma)$ open implies that $\pi^{-1}(\gamma) \subset \operatorname{Int} \overline{B_i^{\nu}}$ and since $\pi(B_i^{\nu})$ and B_i are generically equal we get that $\gamma \subset \operatorname{Int} \overline{B_i}$. It follows that $B_i' = B_i \cup \gamma$ is open, cf. Remark 2.1. Also, B_i' is, obviously, a generically basic set so that $\#(F \cap \widetilde{B_i'}) \neq 3$ for any 4-element fan $F \subset \operatorname{Spec}_r \mathcal{O}(X)$. Thus, it only remains to check that B_i' does not intersect its Zariski boundary. But if $\gamma' \subset \operatorname{Bd}(B_i)$ then $\pi^{-1}(\gamma')$ will intersect $\overline{B_i^{\nu}}$, against the hypothesis. □

Example 3.2. Consider the irreducible analytic set X_0 in $\mathbb{R}^3$ defined by $y^2z^2 = x^2 + y^4$ (see Fig. 14) and its normalization $z'^2 = x'^2 + y'^2$, where $z' = z$, $y' = y$ and $x = x'y'$ (see Fig. 15). Let γ and γ' be, respectively, the positive and negative z-axis of X_0. We have $\pi^{-1}(\gamma) = \gamma_1 \cup \gamma_2$ and $\pi^{-1}(\gamma') = \gamma_1' \cup \gamma_2'$.

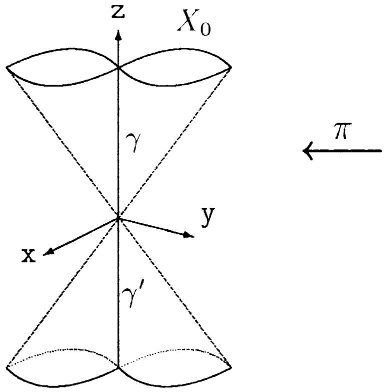

Fig. 14.

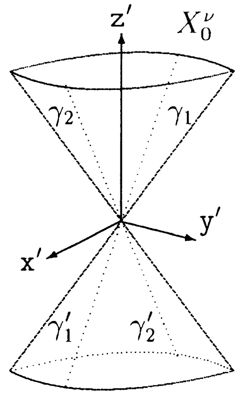

Fig. 15.

If $B_1^\nu = \{(x'-z')^2 - 4(x'+z')^2 - 4y'^2 > 0, z' > 0\}$ (the first equation defines a cone around γ_1), then $B_1 = \{y^2[(x-yz)^2 - 4(x+yz)^2 - 4y^4] > 0, y^2z > 0\}$. Of course, being π an isomorphism out of a set of dimension 1, we have that $\pi(B_1^\nu)$ and B_1 (resp. B_1^ν and $\pi^{-1}(B_1)$) are generically equal in X_0 (resp. X_0^ν). Concerning Lemma 3.1, we have that $B_1^\nu \cup \pi^{-1}(\gamma)$ is not basic open ($\gamma_2 \not\subset \operatorname{Int}\overline{B_1^\nu}$) so we cannot assure $B_1 \cup \gamma$ is basic open and, in fact, it is not for B_1 contains only two of the four sheets around γ and so $\gamma \not\subset \operatorname{Int}\bar{B}_1$.

If $B_2^\nu = \{[(x'-z')^2 - 4(x'+z')^2 - 4y'^2][4y'^2 + 4(x'-z')^2 - (x'+z')^2] > 0, z' > 0\}$ then $B_2 = \{y^4[(x-zy)^2 - 4(x+yz)^2 - 4y^4][4y^4 + 4(x-yz)^2 - (x+yz)^2] > 0, y^4z > 0\}$. Now, $B_2^\nu \cup \pi^{-1}(\gamma)$ is basic open (in fact, $\gamma_1, \gamma_2 \subset B_2^\nu$) and $B_2^\nu \cup \pi^{-1}(\gamma') = (\gamma_1 \cup \gamma_2) \cap B_2^\nu = \emptyset$, so $B_2 \cup \gamma$ is basic open, cf. Lemma 3.1. In this case B_2 contains the four sheets around γ so, this time, $\gamma \subset \operatorname{Int}\bar{B}_2$ and $\gamma' \cap \overline{B_2} = \emptyset$.

We need the following generalization of Proposition 2.7 in order to add several half-branches at once:

Proposition 3.3. *Let X_0 be a normal analytic set germ, B_1 and B_2 basic open and $\gamma_1, \dots, \gamma_r \subset \operatorname{Int}(\overline{B_1 \cup B_2})$ independent half-branches (that is, $\bar{\gamma}_i^Z \cap \bar{\gamma}_j^Z = \emptyset$ if $i \neq j$). Then there exist B_1', B_2' basic open such that $B_1' \cup B_2' = B_1 \cup B_2 \cup \{\bigcup_1^r \gamma_i\}$.*

Moreover, if S is a closed semianalytic set germ such that $(\bigcup_1^r \bar{\gamma}_i^Z) \cap S = \emptyset$, then we can take B_1', B_2' such that $B_j' \cap S = B_j \cap S$, $j = 1, 2$. Also, for each i, if $\bar{\gamma}_i^Z = \gamma_i \cup \gamma_i'$ then $\gamma_i \subset B_1'$, $\gamma_i' \not\subset \overline{B_1'}$.

Proof. We proceed by induction, the case $r = 1$ being Proposition 2.7. Now, suppose we have B_1^k, B_2^k such that $B_1^k \cup B_2^k = B_1 \cup B_2 \cup \{\bigcup_1^k \gamma_i\}$ and $\gamma_i \subset B_1^k$, $\gamma_i' \not\subset \overline{B_1^k}$. Let U_γ and $U_{\gamma'}$ be connected open semianalytic neighborhoods of γ and γ', respectively, such that $\overline{U_\gamma} \cap [S \cup \{f_1^k g_1^k f_2^k g_2^k = 0\}] \subset \gamma$ and $\overline{U_{\gamma'}} \cap [S \cup \{f_1^k g_1^k f_2^k g_2^k = 0\}] \subset \gamma'$ and define $T = X_0 \setminus (U_\gamma \cup U_{\gamma'})$. Applying Proposition 2.7 to B_1^k, B_2^k and T we obtain B_i^{k+1} such that $B_i^{k+1} \cap T = B_i^{(k)} \cap T$. Since $S \subset T$ we have $B_i^{k+1} \cap T = B_i^k \cap T = \cdots = B_i \cap S$ and also, as $\bar{\gamma}_i^Z \cap T = \emptyset$ for $i = 1, \ldots, k$, we have $\gamma_i \subset B_1^{k+1}$, $\gamma_i' \not\subset \overline{B_1^{k+1}}$. $\square$

Corollary 3.4. *If* $\bar{s} = 2 \Rightarrow t = 2$.

Proof. Let C be an open semianalytic set and write $C^\nu = \pi^{-1}(C) = B_1^\nu \cup B_2^\nu$. Let us call Σ to the union of those half-branches γ of X_0 such that $\pi^{-1}(\gamma)$ is not a single half-branch. If $\bar{\gamma}_i^Z = \gamma_i \cup \gamma_i'$, we will have $C \cap \Sigma = \{\gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \gamma_{r+1}', \ldots, \gamma_s, \gamma_s'\}$. As $\bar{s} = 2$ the half-branches in $\pi^{-1}(\gamma_i)$ are independent and by the preceding proposition we can suppose $C^\nu = B_1^\nu \cup B_2^\nu$, with $\pi^{-1}(\gamma_i) \subset B_1^\nu$, $\pi^{-1}(\gamma_i') \not\subset \overline{B_1^\nu}$ for $i = 1, \ldots, r$. Therefore, $B_1 \cup \{\bigcup_1^r \gamma_i\}$ is basic open (see Lemma 3.1). Now, as it must be $\{\gamma_{r+1}, \gamma_{r+1}', \ldots, \gamma_s, \gamma_s'\} \subset \operatorname{Int}(\overline{B_1 \cup B_2})$ (because they are in the open set C which is generically equal to $B_1 \cup B_2$), by Lemma 2.3 we can add a neigborhood of those half-branches to B_1.

Finally, we add all the remaining half-branches (for they are the image of a single half-branch of X_0^ν). For instance, if γ is one of those half-branches and $\bar{\gamma}^Z = \gamma \cup \gamma'$, then by applying Proposition 2.7 to B_1^ν, B_1^ν and $\pi^{-1}(\gamma)$ we can suppose $\pi^{-1}(\gamma) \subset B_1^\nu$, $\pi^{-1}(\gamma') \not\subset \overline{B_1^\nu}$ and, by Lemma 3.1, $B_1 \cup \gamma$ will be basic open. Thus it can be written $C = B_1 \cup B_2$ concluding $t = 2$. $\square$

Thus we can state the central result

Theorem 3.5. *Let X_0 be an analytic set germ of dimension two and $\pi : X_0^\nu \to X_0$ its normalization. Then the following conditions are equivalent*:

(1) *There exist an irreducible analytic curve germ $Y \subset X_0^\nu$ such that $\pi(Y)$ is a unique half-branch.*
(2) $\bar{s}(X_0) = 3$.
(3) *There exists a* 4-*element fan* $F \subset \operatorname{Spec}_r \mathcal{O}(X_0)$ *specializing to a unique ordering* $\tilde{\tau}$ $(F \to \tilde{\tau})$ *with* $\dim \tilde{\tau} = 1$.
(4) $t(X_0) = 3$.

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